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Magnetic Field Effects on Spacetime Around a Magnetized Spherical Star

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Abstract We investigate the effects of magnetic field on the background spacetime of a spherically symmetric relativistic star. Using the general relativistic Maxwell equations coupled to the Einstein field equations for the gravitational field, it is shown that not only the backreaction of the spacetime modifies the magnetic field of the star, but also the magnetic field of the star molds the spacetime in its vicinity. The part played by the poloidal as well as the toroidal components of the magnetic field on the exterior spacetime are investigated.

Keywords Einstein equations · General relativistic Maxwell equations · Magnetic field · Spacetime · Relativistic stars · Magnetars

1 Introduction

Gravitationally bound relativistic systems, such as neutron stars and in particular magnetars, posses very strong magnetic fields [1]. In such cases the magnetic field is modified by the background spacetime [2–5], whereas the spacetime around the star is determined by the matter distribution inside the star. In spherically symmetric stars the spacetime curvature is directly dependent on the total mass density hence energy content of the star. However for gravitational sources with strong magnetic fields the energy density is modified by the presence of the magnetic field [6, 7]. which in turn effects the spacetime. Thus any physical process occurring in vicinity of these objects is effected by the interplay between the background spacetime and the modified magnetic field of the star.

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Here we investigate the effects of magnetic field on the spacetime around a spherically symmetric highly magnetized relativistic star. To incorporate the effects of background spacetime on the magnetic field we study the problem within the framework of coupled Einstein-Maxwell equations. The magnetic field outside the star is determined by the general relativistic Maxwell equations, which is then used to solve the Einstein field equations supplemented with the stress energy-momentum tensor. The resulting equations exhibit a strong coupling between the spacetime metric and the magnetic field of the star. Solving the nonlinear equations numerically it is found that the usual Schwarzschild spacetime is modified by the magnetic energy density of the star for toroidal or poloidal magnetic fields. In the exterior region toroidal magnetic field enhances the metric component hence increases the gravitational field of the star. On the other hand in the absence of a toroidal magnetic field, the non-vanishing poloidal magnetic field results in reducing the gravitational field strength. In this case the spacetime curvature is less than for the Schwarzschild spacetime, however the variation in the metric component is enhance especially close to the gravitational source.

The paper is organized as follows. In Sect. 2 we formulate the problem in a general relativistic framework. Assuming a spherically symmetric form of the metric, the metric components are the determined by explicitly formulating the Einstein field equations. The magnetic field is then coupled to the gravitational field equations by using the energy-momentum tensor for the electromagnetic field. Here the components of the energy-momentum tensor are calculated from the electromagnetic field tensor and also the general relativistic Maxwell equations. In Sect. 3 the spacetime around the spherical star is determined by solving the resulting nonlinear equation numerically. We interpret and discuss these results in the concluding section (Sect. 4). Throughout this paper we employ the gravitational units $G = 1 = c$, unless mentioned otherwise.

2 The Einstein-Maxwell Equations

The general form of the spacetime around a spherically symmetric mass is given by the metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

According to the general theory of relativity the explicit form of the metric components in the exterior region $r > R$ is determined by the Einstein field equations,

$$R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R = 8\pi T^\alpha_\beta, \tag{2}$$

Here the left side of the equation is the Einstein tensor representing the spacetime curvature and on the right hand side we have the stress energy-momentum tensor.

For a spherical star possessing an electromagnetic field, the stress energy-momentum tensor is given by [8]

$$T^\alpha_\beta = \frac{1}{4\pi}\left(F^{\alpha\gamma}F_{\gamma\beta} - \frac{1}{4}\delta^\alpha_\beta F_{\mu\nu}F^{\mu\nu}\right), \tag{3}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor:

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}. \tag{4}$$

The explicit form of the electromagnetic field tensor is given by the general relativistic Maxwell equations

$$(\sqrt{-g}F^{\alpha\beta})_{,\beta} = 4\pi\sqrt{-g}J^{\alpha}. \tag{5}$$

where J^{α} is current four vector. In general the current four vector is the sum of two terms corresponding to convection and conduction currents:

$$J^{\alpha} = \epsilon u^{\alpha} + \sigma u_{\beta}F^{\beta\alpha}, \tag{6}$$

where ϵ is the proper charge density and σ is the electrical conductivity of the gas. For the spherically symmetric spacetime the orthonormality condition $u^{\alpha}u_{\alpha} = -1$ gives for the 4-velocity vector,

$$u^{\alpha} = (e^{-\nu/2}, 0, 0, 0), \qquad u_{\alpha} = (-e^{\nu/2}, 0, 0, 0). \tag{7}$$

For highly magnetized stars the electrical charge on the star is small hence the result electric field when compared with the magnetic field is negligible. For radius dependent magnetic field this implies that the radial component of the magnetic field, being curl of the vector potential, must vanish. The gravitational field equations (2) coupled to the Maxwell equations (5) then give, for non-vanishing, non-identical components, the following set of determining equations

$$\frac{e^{-\lambda}}{r^2}(1 - r\lambda') - \frac{1}{r^2} = f(r), \tag{8}$$

$$\frac{e^{-\lambda}}{r^2}(1 + r\nu') - \frac{1}{r^2} = f(r), \tag{9}$$

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu' - \lambda'}{2r} + \frac{\nu'}{4}(\nu' - \lambda')\right] = g(r) \tag{10}$$

$$\frac{e^{-\lambda}}{r^2}B^2B^3 = 0, \tag{11}$$

where

$$f(r) = \left[\frac{e^{-\lambda}}{r^2}\{(B^3)^2 + (B^2)^2\}\right], \tag{12}$$

$$g(r) = \left[\frac{e^{-\lambda}}{r^2}\{(B^3)^2 - (B^2)^2\}\right], \tag{13}$$

and the prime denotes differentiation with respect to r.

3 The Exterior Spacetime and the Modified Magnetic Field

To obtain the explicit dependence of the metric components $e^{\nu(r)}$ and $e^{-\lambda}$ on the magnetic field components, we first notice that a subtraction of (8) and (9) gives $\nu' = -\lambda'$. As in the Schwarzschild spacetime, in the asymptotic limit the constant of integration here must vanish and we obtain $e^{\nu(r)} = e^{-\lambda(r)}$ for all values of r.

Further in view of (11) we must take either the toroidal component of the magnetic field B^3 or the poloidal component B^2 to be non-vanishing. In the following we discuss these cases separately.

3.1 Non-vanishing Toroidal Magnetic Field

This gives in (12) $f(r) = g(r)$ to be $e^{-\lambda}(B^3)^2/\,r^2$. Using (4) we see that the electromagnetic field tensor has the nonvanishing components $F_{12} = -F_{21}$ and $F_{13} = -F_{31}$. The Maxwell equation (5) gives for the nonvanishing component B^3 in the exterior region:

$$B^3(r) = B_0 e^{\lambda(r)}. \tag{14}$$

where B_0 is a constant. We notice here that the magnetic field is determined by the spacetime metric.

Substitution into the Einstein field equation (10) we obtain

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu' - \lambda'}{2r} + \frac{\nu'}{4}(\nu' - \lambda')\right] = \frac{e^{-\lambda}(B^3)^2}{r^2}. \tag{15}$$

Using (14) for the magnetic field and $\nu = -\lambda$ this simplifies to

$$\frac{d^2}{dr^2}\left(re^{-\lambda}\right) = \frac{2B_0^2 e^{\lambda}}{r}. \tag{16}$$

3.2 Non-vanishing Poloidal Magnetic Field

In this case we have in (13) $g(r) = -f(r)$ to be $-e^{-\lambda}(B^2)^2/\,r^2$. Correspondingly the Maxwell equation (5) gives for the poloidal component:

$$B^2(r) = \tilde{B}_0 e^{\lambda(r)}. \tag{17}$$

where $\tilde{B}_0$ is a constant. We notice here that the magnetic field is determined by the spacetime metric.

Substitution into the Einstein field equation (10) we obtain

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu' - \lambda'}{2r} + \frac{\nu'}{4}(\nu' - \lambda')\right] = -\frac{e^{-\lambda}(B^3)^2}{r^2}. \tag{18}$$

Using (14) and $\nu = -\lambda$ we have

$$\frac{d^2}{dr^2}\left(re^{-\lambda}\right) = -\frac{2\tilde{B}_0^2 e^{\lambda}}{r}. \tag{19}$$

In both cases the coupling between the magnetic field and the background spacetime gives highly nonlinear equations. To identify the important physical implications of these equations we solve (16) and (19) numerically. Figure 1 and 2 show plots for the toroidal and the poloidal components with varying magnetic field strengths; where we have $e^{-\lambda} = 1$, $de^{-\lambda}/dr = 10^3$ close to the singularity at zero radial distance (here at $r = 10^{-3}$). Also we have employed the gravitational units in both cases.

4 Discussion and Conclusions

We have studied the spacetime around a spherically symmetric star endowed with a strong magnetic field. The effects of magnetic field (energy density) on the spacetime in the stellar vicinity are included by a coupling between the Einstein field equations to the general

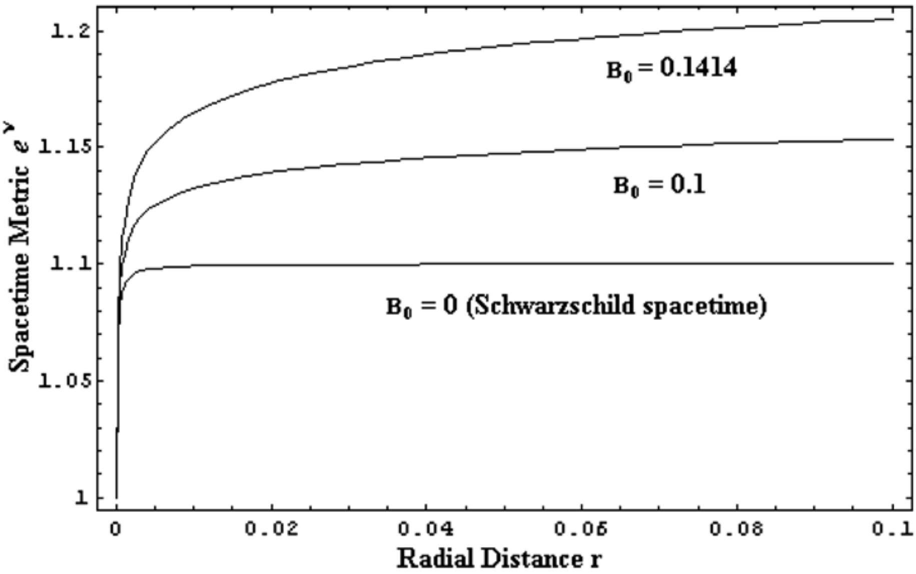

Fig. 1 The spacetime metric $e^{\nu(r)}$ for the region close to the star, with different values of the magnetic field strength of the toroidal component

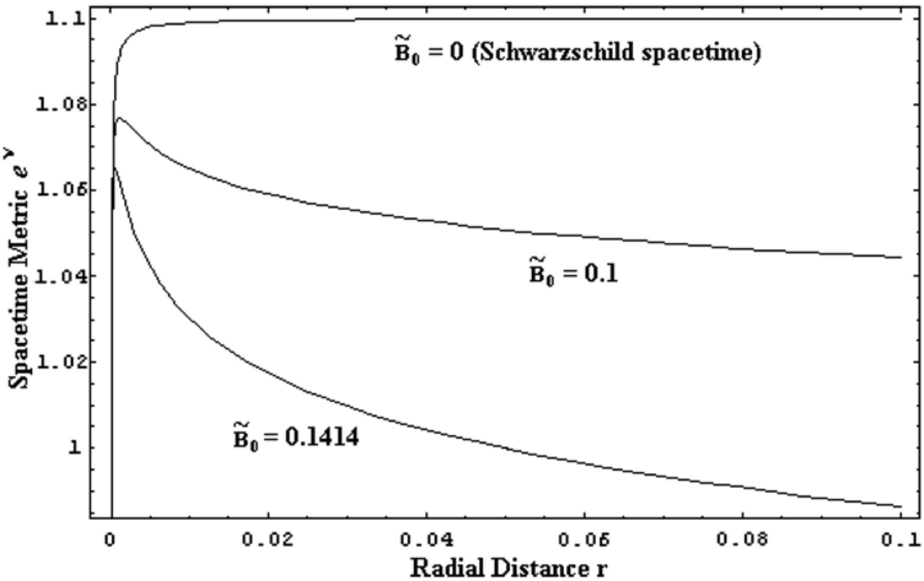

Fig. 2 The spacetime metric $e^{\nu(r)}$ for the region close to the star, with different values of the magnetic field strength of the poloidal component

relativistic Maxwell equations via the stress energy-momentum tensor for the field. The resulting equations (16) and (19) show that there is a strong nonlinear coupling between the spacetime metric and the magnetic field. Thus not only the spacetime determines the magnetic field around the star, but also the spacetime itself is determined by the magnetic energy density in the stellar vicinity. Furthermore for a radial dependence of the magnetic field, the analysis shows that in the case of a toroidal magnetic field the poloidal field must vanish and vice-versa. An increase in the toroidal magnetic field strength enhances the metric component (Fig. 1) hence close to the star the spacetime is more curved than in the Schwarzschild case. The effect of a non-zero poloidal magnetic field (Fig. 2) is however more complicated. Here the higher poloidal magnetic field results reducing the gravitational field strength, hence the spacetime curvature is less than for the Schwarzschild spacetime. However there is a higher variation in the metric component, especially close to the singular point at $r = 0$.

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